The sizes of individuals, x, are distributed according to $p(x)\mathrm{d}x$, the probability that an individual has size between x and $x + \mathrm{d}x$. $q(x) = \int_{-\infty}^{x} p(y)\mathrm{d}y$ is the cumulative probability distribution; that is, the probability that a random individual has size $\leq x$, or the fraction of the population of size $\leq x$. The mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.

The Lorenz curve $L(f)$ is given by

$$L(f) = \frac{1}{\mu} \int_{q(x)=0}^{f} x\mathrm{d}q(x)$$

in which f represents the *fraction of the total population*; clearly, $f = q(y)$ where y is the size of the "f^{th} individual". Note that $L(f) \leq f$, with equality only when $p(x) =$ const. $L(f) = f$ is thus the Lorenz curve of a uniform distribution and is called the "line of perfect equality".

The mean difference between pairs of individuals is given by:

$$g = \frac{1}{2} \int_{x} \int_{y} p(x)p(y)|x - y|\mathrm{d}x\mathrm{d}y$$

(1/2 otherwise we will count each pair twice) or, by considering splitting the $x - y$ plane into two identical triangles,

$$g = \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{x} p(y)(x - y)\mathrm{d}y\mathrm{d}x.$$

Split the integral up, giving,

$$g = \int_{-\infty}^{\infty} xp(x)q(x)\mathrm{d}x - \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{x} yp(y)\mathrm{d}y\mathrm{d}x$$

Consider the second integral. Observe that

$$\frac{\mathrm{d}q(y)}{\mathrm{d}y} = p(y), \quad \text{so}, \quad \mathrm{d}q(y) = p(y)\mathrm{d}y,$$

so that we can write

$$\int_{-\infty}^{x} yp(y)\mathrm{d}y = \int_{0}^{q(x)} q(y)\mathrm{d}q(y) = \mu L(q(x))$$

giving

$$g = \int_{-\infty}^{\infty} xp(x)q(x)\mathrm{d}x - \int_{-\infty}^{\infty} p(x)L(q(x))\mathrm{d}x$$

Performing the same change of variables and writing $f = q(x)$,

$$g = \int_{-\infty}^{\infty} xp(x)q(x)\mathrm{d}x - \mu \int_{0}^{1} L(f)\mathrm{d}f,$$

since $q(-\infty) = 0$ and $q(\infty) = 1$.

Now consider the first integral. By parts,

$$\int_a^b v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x = [uv]_a^b - \int_a^b u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x$$

Identify $v = q(x)$, $\mathrm{d}v/\mathrm{d}x = p(x)$; $\mathrm{d}u/\mathrm{d}x = xp(x)$, $u = \mu L(q(x))$. Now,

$$\int_{-\infty}^{\infty} xp(x)q(x)\mathrm{d}x = [\mu L(q(x))q(x)]_{-\infty}^{\infty} - \mu \int_{-\infty}^{\infty} p(x)L(q(x))\mathrm{d}x$$

and therefore,

$$\int_{-\infty}^{\infty} xp(x)q(x)\mathrm{d}x = \mu - \int_0^1 L(f)\mathrm{d}f$$

so that

$$g = \mu \left(1 - 2\int_0^1 L(f)\mathrm{d}f\right)$$

Compare this to the Gini coefficient, which is defined as "the area between the Lorenz curve and the line of perfect equality, relative to the whole area below the line of perfect equality", or,

$$G = 2\int_0^1 f - L(f)\mathrm{d}f$$

Clearly $G = g/\mu$, as required.